

The University of Chicago

ON SPECIAL JORDAN ALGEBRAS

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS
JUNE, 1942

By
GERHARD KARL KALISCH

CHICAGO, ILLINOIS

1945

The University of Chicago

ON SPECIAL JORDAN ALGEBRAS

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS
JUNE, 1942

By
GERHARD KARL KALISCH

CHICAGO, ILLINOIS

1945



TABLE OF CONTENTS

Section	Page
1. Introduction	1
2. Involutions	2
3. Enveloping algebras of subspaces	4
4. The simplicity of the algebra S^0	6
5. The algebras S_J	8
6. Extension of isomorphisms	10
7. Classification of the algebras S_J over special fields	18
8. The main theorem	19



Digitized by the Internet Archive
in 2026

https://archive.org/details/IA41533916_0080

ON SPECIAL JORDAN ALGEBRAS

1. Introduction. Let S be an associative algebra of finite order over any field F of characteristic not two and ab represent the product of any two quantities a and b of S . Then the operation

$$a \cdot b = \frac{1}{2} (ab + ba) \quad (a, b \text{ in } S)$$

is called quasimultiplication¹, and any linear subspace A over F of S , which is closed with respect to this operation, forms a corresponding algebra A . We call any algebra isomorphic to such an algebra a special Jordan algebra and see that special Jordan algebras are commutative but not, in general, associative.

It is the purpose of this paper to begin a study of simple special Jordan algebras. Any associative algebra S forms a special Jordan algebra S^0 with respect to quasimultiplication and we shall show that if S is simple so is S^0 . If S has an involution J and S_J is the set of all J -symmetric quantities of S , the set S_J is a special Jordan algebra under quasimultiplication. Then we shall show again that S_J is simple if S is.

Our study is analogous to that which has already been made for Lie^2 algebras and we shall show, as in that theory, that

¹Cf. P. Jordan, Zeitschrift fuer Physik, 80, (1933), p. 285; Gottinger Nachrichten, (1932), p. 569; (1933), p. 209; P. Jordan, J. v. Neumann, E. Wigner, Annals of Mathematics, 35 (1934), p. 29. In the first three papers Jordan considered the general class of all algebras satisfying the property $x(yx^2) = (xy)x^2$ for all x and y . In the final one of these papers the assumption of reality and commutativity was made. We shall consider only the commutative case.

²Cf. N. Jacobson, Annals of Mathematics, 38 (1937), p. 508;

if K is a scalar extension of F such that A_K is a special Jordan algebra of one of the types above then A is a special Jordan algebra over F of a corresponding type. The corresponding analysis for Lie algebras formed a major part of the determination of all simple Lie algebras over any field of characteristic not two and it is expected that our study will occupy a corresponding place in the theory of Jordan algebras.

2. Involutions. An involution of an associative algebra S over F is a linear transformation J over F of S such that $J^2 = I$, $(ab)^J = b^J a^J$ for every a and b of S . The quantities $a = a^J$ of S are called J -symmetric and span a linear subspace S_J of S . The quantities $a = -a^J$ of S are called J -skew and span a linear subspace S_J^S of S such that S is the supplementary sum

$$S = S_J + S_J^S.$$

We now prove

LEMMA 1. Let S be an associative algebra over F , J be an involution over F of S , K be a scalar extension field of F . Then J can be extended³ uniquely to an involution over K (which we shall also designate by J) of S_K and

$$(S_K)_J = (S_J)_K.$$

For any linear transformation J of a linear space S of order n over F is uniquely determined, with respect to a fixed basis of S over F , by an n -rowed matrix with elements in F and

ibid., 39 (1938), p. 181; Duke Mathematical Journal, 3 (1938), p. 534; W. Landherr, Hamburger Abhandlungen, 11 (1935), p. 41; Abhandlungen der Hansischen Universitaet, 12 (1938), p. 200.

³Cf. N. Jacobson, A class of normal simple Lie algebras of characteristic zero, Annals of Mathematics, 38 (1937), p. 509.

this same matrix uniquely determines the extension of J to S_K . It is easily seen that the extension is an involution of S_K and it is clear that $(S_J)_K \subseteq (S_K)_J$. If a is in S_K then $a = a_1 \xi_1 + \dots + a_r \xi_r$ where $\xi_1, \dots, \xi_r$ are in K and have the property that $x_1 \xi_1 + \dots + x_r \xi_r = 0$ for the x_i in S if and only if the x_i are all zero. Then if a is in $(S_K)_J$ we have $a - a^J = \sum (a_i - a_i^J) \xi_i = 0$ where the a_i^J are in S , $a_i - a_i^J = 0$, the a_i are in S_J , a is in $(S_J)_K$, $(S_K)_J \subseteq (S_J)_K$, $(S_K)_J = (S_J)_K$ as desired.

We shall be principally interested in the case where S is a simple algebra and for such an algebra we shall use the known⁴ result which we state as

LEMMA 2. Let S be a simple associative algebra over F , J_0 and J be involutions over F such that $k^J = k^{J_0}$ for every k in the centrum of S . Then

$$(1) \quad J = J_0 T$$

where T is the inner automorphism

$$(2) \quad a \mapsto a^T = t a t^{-1}$$

of S defined for a regular J_0 -symmetric or J_0 -skew quantity t of S .

Let us also observe

LEMMA 3. The set $S_J = S_{J_0} t^{-1}$ or $S_{J_0}^s t^{-1}$ according as t is J_0 -symmetric or J_0 -skew.

For $a = a^J$ is in S_J if and only if $a = a^{J_0 T} = t a^{J_0} t^{-1}$, $a t = t a^{J_0}$, $(a t)^{J_0} = a t^{J_0} = t a t$. Thus $a t$ is in S_{J_0} or $S_{J_0}^s$ respectively as desired.

The result of Lemma 2 implies also

LEMMA 4. Let $J = J_0 T$, $K = J_0 U$ for inner automorphisms T

⁴See A. A. Albert, Structure of Algebras, p. 154.

and U of a simple algebra S defined by corresponding J_0 -symmetric or J_0 -skew quantities t and u . Then J and K are cogredient in F if and only if t is J_0 -congruent to αu for α in F .

For J and K are cogredient if and only if $J = R^{-1}KR$ for an automorphism R over F of S . Then R is an inner automorphism $a \rightarrow rar^{-1}$ and $a^J = a^{J_0T} = a^{R^{-1}J_0UR} = (r^{-1}ar)^{J_0UR} = [r^{J_0}a^{J_0}(r^{J_0})^{-1}]^{UR}$. Replace a^{J_0} by a and have $a^T = [r^{J_0}a(r^{J_0})^{-1}]^{UR} = (rur^{J_0})a(rur^{J_0})^{-1} = tat^{-1}$ for every a in S if and only if $rur^{J_0} = \alpha t$ for α in F .

3. Enveloping algebras of subspaces. If B is any subset of an algebra S over F we define its closure $\bar{B}$ to be the intersection of all subalgebras of S containing B . Then $\bar{B}$ is the (enveloping) subalgebra of S consisting of all finite polynomials over F in the quantities of B .

If A is any linear subspace of an associative algebra S and T is an automorphism $a \rightarrow a^T$ of S the linear space A^T of all a^T for a in A has the same order as A and if A is a special Jordan algebra A^T is clearly a special Jordan algebra isomorphic to A . The enveloping algebras $\bar{A}$ and $\bar{A}^T$ are isomorphic associative algebras and if $\bar{A} = S$ then $\bar{A}^T = S$. If $A = S_J$ and K is an involution cogredient to J then $K = T^{-1}JT$ for an automorphism T of S and if $a = a^J$ the quantity $(a^T)^K = a^{TT^{-1}JT} = a^T$. Hence $S_K = (S_J)^T$. Thus we have

LEMMA 5. If J and K are cogredient the corresponding special Jordan algebras S_J and S_K are isomorphic and their enveloping associative algebras $\bar{S}_J$ and $\bar{S}_K$ are isomorphic.

We now consider a special case and prove

LEMMA 6. Let J be the transformation of transposition on a total matrix algebra S of degree n . Then $\bar{S}_J = S$.

If $n = 1$ we have $S_J = S = \overline{S_J}$. Let $n > 1$ and $e_{11}, \dots, e_{ij}, \dots, e_{nn}$ be an ordinary matrix basis of S and thus $e_{ij}^J = e_{ji}$. Then every e_{ii} and $e_{ij} + e_{ji}$ is in S , $e_{ii}(e_{ij} + e_{ji}) = e_{ij}$ is in S_J for every $i \neq j$. Hence S_J contains our basis of S and $\overline{S_J} = S$.

Our next special case will be that of a total matrix algebra S of degree $n = 2m$ over F . Then $S = M \times N$ where M has degree m and an ordinary matrix basis $e_{11}, \dots, e_{ij}, \dots, e_{mm}$, N has degree two and an ordinary matrix basis $g_{11}, g_{12}, g_{21}, g_{22}$. Then every quantity of S is uniquely expressible in the form

$$(3) \quad a = Ag_{11} + Bg_{12} + Cg_{21} + Dg_{22}$$

for A, B, C, D in M and the transformation of transposition in S may be chosen so that

$$(4) \quad a' = A'g_{11} + C'g_{12} + B'g_{21} + D'g_{22}.$$

The unity quantity I of S is the unity quantity $g_{11} + g_{22}$ of both M and N and every nonsingular skew quantity t of S is congruent to $t = g_{12} - g_{21}$. Define

$$(5) \quad a^J = ta't^{-1}$$

and see that S_J consists of all quantities (3) with

$$(6) \quad D = A', \quad B' = -B, \quad C' = -C.$$

Clearly the intersection of S_J and M is the set of $A = A'$ in M .

By Lemma 6 the algebra S_J contains M . If $m = 1$ the skew matrices B and C are zero, every $A' = A$, $S_J = M$ has order one and $\overline{S_J} \neq S$. If $m > 1$ we see that $\overline{S_J}$ contains $e_{jj}[e_{ji} - e_{ij}]g_{12} = e_{ji}g_{12}$ for $i \neq j$, as well as $e_{ij}(e_{ji}g_{12}) = e_{ii}g_{12}$ for every i . Hence $\overline{S_J}$ contains $g_{12} = \sum_i e_{ii}g_{12}$. Similarly $\overline{S_J}$ contains g_{21} , $g_{12}g_{21} = g_{11}$, $g_{21}g_{12} = g_{22}$, $\overline{S_J} = S$.

If J is an involution over F of any simple algebra S of degree n over its centrum F then there exists a splitting field K (of finite degree over F) of S such that S_K is a total matrix algebra. By Lemma 1 we may regard J as being defined in S_K . Using Lemma 2 we have $a^J = ta't^{-1}$ where $t = it'$ is in S_K . If K_1 is any other splitting field of S and $a^J = t_1a't_1^{-1}$ the quantities t and t_1 differ by a factor in the composite of K and K_1 and t_1 is symmetric or skew according as t is symmetric or skew. Let us then call J symmetric or skew in the respective cases.

If J is symmetric there exists a splitting field K of S such that t is congruent in K to the unity quantity of S_K , J is cogredient in K to transposition. Then $(S_J)_K = (S_K)_J$ is isomorphic to the set of all symmetric matrices over K an algebra of order $1/2n(n+1)$. S_J has order $1/2n(n+1)$. Also by Lemma 6 $\overline{(S_K)_J} = S_K$, and clearly $\overline{(S_J)_K} = \overline{(S_K)_J} = S_K$, the order of $\overline{S_J}$ is the order of S , $\overline{S_J} = S$. If J is skew then n is even, $n = 2m$, $(S_J)_K$ is isomorphic to the set of n -rowed matrices defined in (3), (6). The order of this set is $m^2 + 2(1/2m(m-1)) = 3m^2 - m = 1/2n(n-1)$ and, by the argument above, $\overline{S_J} = S$ unless $m = 1$. We have proved

THEOREM 1. Let S be a simple associative algebra of degree n over its centrum F and J be an involution over F of S . Then the order of the special Jordan algebra S_J is $1/2n(n+1)$ or $1/2n(n-1)$ according as J is symmetric or skew and n is even in the latter case. Also $\overline{S_J} = S$ except when J is skew and S_J has order one.

4. The simplicity of the algebra S^0 . The principal result on the special Jordan algebra S^0 obtained by quasimultiplication from an associative algebra S is

THEOREM 2. The special Jordan algebra S^0 is simple if and only if S is simple.

If B is an ideal of S the products ab and ba are in B for every b of B and a of S , $1/2(ab + ba)$ is also in B , B^0 is an ideal of S^0 . If S^0 is simple the linear space B^0 is zero or S^0 , B is zero or S , S is simple.

Conversely let S be simple. Then it is clear that not only S but also S^0 may be regarded as an algebra over the centrum of S and we may assume that S is a normal simple algebra over F . Suppose first that S is a total matric algebra and that $e_{11}, \dots, e_{ij}, \dots, e_{nn}$ form an ordinary matric basis of S . Then if B is a nonzero ideal of S^0 and $b = \sum_{i,j} \beta_{ij} e_{ij} \neq 0$ is in B so is $e_{11}b + be_{11} = \sum_j \beta_{1j} e_{1j} + \sum_k \beta_{k1} e_{k1}$. Also $e_{jj}(e_{11}b + be_{11}) + (e_{11}b + be_{11})e_{jj} = \beta_{j1} e_{j1} + \beta_{1j} e_{1j}$ is in B for every i and j . If some $\beta_{11} \neq 0$ the ideal B contains $2\beta_{11} e_{11}$ and hence e_{11} . Otherwise some $\beta_{1j} \neq 0$ for $i \neq j$, $b_0 = e_{1j}(\beta_{j1} e_{j1} + \beta_{1j} e_{1j}) + (\beta_{j1} e_{j1} + \beta_{1j} e_{1j})e_{1j} = \beta_{j1} e_{11} + \beta_{1j} e_{jj}$ is in B and by the process above we have a quantity e_{11} in B . Then $e_{11}e_{1j} + e_{1j}e_{11} = e_{1j}$ if $i \neq j$ and hence B contains every e_{1j} for our fixed i . Similarly B contains $e_{j1}e_{11} + e_{11}e_{j1} = e_{j1}$ and $e_{1j}e_{j1} + e_{j1}e_{1j} = e_{11}$. It follows that B contains every e_{jj} and, by the process above every e_{ij} , $B = S^0$, S^0 is simple. If now S is any simple algebra over its centrum F there exists a scalar extension K of F such that S_K is a total matric algebra and hence $(S_K)^0$ is normal simple. But $(S_K)^0 = (S^0)_K$ is then simple and is indeed normal simple, S^0 is normal⁵ simple. This proves our theorem.

COROLLARY. The algebra S^0 is normal simple if and only if S is normal simple.

⁵An algebra A is normal simple if A_K is simple for every scalar extension K of F .

5. The algebras S_J . Let S be a total matrix algebra of degree n and J be the transformation of transposition so that S_J has a basis consisting of the quantities

$$(7) \quad f_{ij} = f_{ji} = \frac{1}{2}(e_{ij} + e_{ji}) \quad (i, j = 1, \dots, n).$$

Multiplication in the commutative algebra S_J is defined by

$$(8) \quad \begin{aligned} f_{11}^2 &= f_{11}, & 4f_{ij}^2 &= f_{11} + f_{jj}, & 4f_{ij} \cdot f_{ik} &= f_{jk}, \\ 2f_{11} \cdot f_{ik} &= f_{ik}, & f_{11} \cdot f_{jj} &= f_{11} \cdot f_{tk} = f_{ij} \cdot f_{tk} &= 0 \end{aligned}$$

for i, j, k, t all distinct and ranging from 1 to n . If B is an ideal of S_J and $b = \sum_{i,j} \beta_{ij} f_{ij}$ is in B so are $2f_{11} \cdot b = \sum_{i \neq j} \beta_{ij} f_{ij} + 2\beta_{11} f_{11}$ and $(2f_{11} \cdot b) \cdot (2f_{jj}) = \beta_{ij} f_{ij}$ for $i \neq j$. If all $\beta_{ij} = 0$ then there is some $\beta_{ii} \neq 0$, $b = \sum \beta_{jj} f_{jj}$, f_{11} is in B for some i , and $2f_{11} \cdot f_{ij} = f_{ij}$ is in B . Hence if $B \neq 0$ there is some f_{ij} in B , $(2f_{ij})^2 = f_{11} + f_{jj}$ is in B and $f_{11} = f_{11} \cdot (f_{11} + f_{jj})$ is in B . So is $2f_{11} \cdot f_{ik} = f_{ik}$, $(2f_{ik})^2 = f_{11} + f_{kk}$ and the difference f_{kk} . Then B contains every f_{ii} , B contains every $2f_{11} \cdot f_{ij} = f_{ij}$, $B = S_J$, S_J is simple.

We next let J be defined as in (5) so that if $m = 1$ the algebra S_J is isomorphic to F and is simple. Hence let $m > 1$ and $a = Ag_{11} + A'g_{22} + Bg_{12} + Cg_{21}$. Then if G is in M , H and K are skew quantities of M we have the quasiproduts

$$(9) \quad \begin{aligned} (2Gg_{11} + 2G'g_{22}) \cdot a &= (GA + AG)g_{11} + (GA + AG)'g_{22} \\ &\quad + (GB + BG')g_{12} + (G'C + CG)g_{21}, \end{aligned}$$

$$(10) \quad (2Hg_{12}) \cdot a = HAg_{11} + (HC)'g_{22} + (HA' + AH)g_{12},$$

$$(11) \quad (2Kg_{21}) \cdot a = BKg_{11} + (BK)'g_{22} + (KA + A'K)g_{21}.$$

Let B be a nonzero ideal of S_J and $b = A_0g_{11} + A'_0g_{22} + B_0g_{12} + C_0g_{21} \neq 0$ be in B . If $A_0 = 0$ then B_0 and C_0 are not both zero.

If $B_0 \neq 0$ we can write $B_0 = PB_1P'$ where $B_1^2 \neq 0$, $K = P'^{-1}B_0P^{-1} \neq 0$ is skew, $B_0K = PB_1^2P^{-1} \neq 0$. Then we use (11) to see that $(2Kg_{21}) \cdot b$ is in B and the coefficient of $g_{11} \neq 0$. Similarly if $C_0 \neq 0$ we may choose H so that $HC_0 \neq 0$. Hence there is a quantity $b \neq 0$ in B such that $A_0 \neq 0$. Now B contains all $(2Gg_{11} + 2G'g_{22}) \cdot b$ and the coefficient of g_{11} is GA_0 . The set of all such quantities contains A_0 and is a nonzero ideal B_1 of M^0 . By Theorem 2 M^0 is simple, $B_1 = M^0$, there is a quantity

$$(12) \quad d = g_{11} + g_{22} + B_2g_{12} + C_2g_{21}$$

is in B . If $B_2C_2 = I$ we form the quasiproduct $(2B_2g_{12}) \cdot d$ by the use of (10) to get

$$(13) \quad d_0 = g_{11} + g_{22} + 2B_2g_{12},$$

and, by (11), we form $(C_2g_{21}) \cdot d_0$ and obtain

$$(14) \quad d_1 = g_{11} + g_{22} + C_2g_{21}.$$

Then since d_0 and d_1 are in B we have $d_0 - 2(d - d_1) = g_{11} + g_{22}$ is also in B and $B = S_J$ in this case. Hence we may assume

$B_2C_2 \neq I$. Form a quantity $(B_2g_{12} + C_2g_{21}) \cdot d$ by the use of (10) and (11) to get

$$(15) \quad h = B_2C_2g_{11} + (B_2C_2)'g_{22} + B_2g_{12} + C_2g_{21}$$

and we have

$$(16) \quad d - h = A_2g_{11} + A_2'g_{22}$$

in B where $A_2 = I - B_2C_2 \neq 0$. Then B contains all $(GA_2 + A_2G)g_{11} + (GA_2 + A_2G)'g_{22}$ and, as in the proof above, every $Ag_{11} + A'g_{22}$. Using (10) and (11) we see that B contains every Bg_{12} and Cg_{21} , $B = S_J$.

We may now prove

THEOREM 3. The special Jordan algebras S_J of Theorem 1 are normal simple algebras.

For there exists a scalar extension K of the centrum F of S such that S_K is a total matrix algebra, J in S_K is cogredient to one of the involutions J_0 above, $(S_K)_J$ is isomorphic to $(S_K)_{J_0}$ and is simple. If B is an ideal of S_J the space B_K is an ideal of $(S_J)_K = (S_K)_J$ and is zero or $(S_J)_K$, B is zero or S_J , S_J is simple. Since S_K is simple for every scalar extension K of the centrum F of S so is $(S_J)_K$, and our theorem is proved.

6. Extension of isomorphisms. If a linear subspace A of an associative algebra S over F is a special Jordan algebra any automorphism T of S carries each quantity a of A into a quantity a^T such that

$$2(a \cdot b)^T = (ab + ba)^T = (a^T b^T + b^T a^T) = 2a^T \cdot b^T.$$

Hence the linear mapping $a \rightarrow a^T$ is an isomorphism of A and the special Jordan algebra A^T of all the a^T . Moreover in some cases we may prove conversely that if A and B are isomorphic special Jordan subspaces of S then there is an automorphism T of S such that $B = A^T$ and $a \rightarrow a^T$ is the given isomorphism. Evidently our hypothesis is that T is a linear mapping of a prescribed subspace A of S on a second linear subspace A^T such that

$$(17) \quad (ab + ba)^T = a^T b^T + b^T a^T \quad (a, b \text{ in } A),$$

and we wish to define a linear transformation U on S such that

$$(18) \quad a^U = a^T, \quad (bc)^U = b^U c^U$$

for every a of A and every b and c of S . We begin our investigation by proving

LEMMA 7. If e is any idempotent of A the quantity e^T is idempotent. If e and f are orthogonal idempotents so are e^T and f^T .

For by (17) $(e^2)^T = (e^T)^2 = e^T$ if $e^2 = e$. If $ef = fe = 0$ we use (17) to get $ef + fe = 0 = 0^T = e^T f^T + f^T e^T$. Then $e^T(e^T f^T + f^T e^T) = e^T f^T + e^T f^T e^T = (e^T f^T + f^T e^T)e^T = e^T f^T e^T + f^T e^T e^T$, $e^T f^T = f^T e^T = -f^T e^T = 0$ as desired.

We now consider the algebra S_j of (7), (8) and have

$$(19) \quad \begin{aligned} e_{11} &= f_{11}, & 4(f_{1j}^T)^2 &= f_{11}^T + f_{jj}^T, \\ 2(f_{1j}^T f_{ik}^T + f_{ik}^T f_{1j}^T) &= f_{jk}^T, & (f_{11}^T f_{ik}^T + f_{ik}^T f_{11}^T) &= f_{ik}^T, \\ f_{11}^T f_{jj}^T + f_{jj}^T f_{11}^T &= f_{1j}^T f_{tk}^T + f_{tk}^T f_{1j}^T = f_{11}^T f_{tk}^T + f_{tk}^T f_{11}^T = 0 \end{aligned}$$

for $i, j, k, t = 1, \dots, n$ and i, j, k, t all distinct. Define a linear transformation U on S by

$$(20) \quad \begin{aligned} e_{jj}^U &= f_{jj}^T = e_{jj}^T, & e_{ij}^U &= 2f_{11}^T f_{ij}^T f_{jj}^T \\ (i &\neq j; i, j = 1, \dots, n), \end{aligned}$$

so that, by Lemma 7, we have

$$(21) \quad (e_{11}^U)^2 = e_{11}^U \neq 0, \quad e_{11}^U e_{jj}^U = 0 \quad (i \neq j; i, j = 1, \dots, n).$$

Then by (20) we have

$$(22) \quad \begin{aligned} e_{11}^U e_{ij}^U &= e_{ij}^U e_{jj}^U = e_{ij}^U, & e_{ij}^U e_{tk}^U &= 0 \\ (j &\neq t; i, j, t = 1, \dots, n). \end{aligned}$$

We compute $e_{ij}^U e_{jk}^U = 4e_{11}^T f_{1j}^T e_{jj}^T f_{jk}^T e_{kk}^T$ for $j \neq i, k$. Use (19) to obtain

$$(23) \quad e_{jj}^T f_{jk}^T = f_{jk}^T - f_{jk}^T e_{jj}^T,$$

and have $e_{ij}^U e_{jk}^U = 4e_{11}^T f_{1j}^T f_{jk}^T e_{kk}^T$. If $k \neq 1$ we have $2f_{1j}^T f_{jk}^T = f_{ik}^T - 2f_{jk}^T f_{ij}^T$ and since $e_{11}^T f_{jk}^T + f_{jk}^T e_{11}^T = e_{kk}^T f_{1j}^T + f_{1j}^T e_{kk}^T = 0$ for

$i \neq j, i \neq k, k \neq j$ we obtain $e_{ii}^T f_{jk}^T f_{ij}^T e_{kk}^T = f_{jk}^T e_{ii}^T e_{kk}^T f_{ij}^T = 0$,

$$(24) \quad e_{ij}^U e_{jk}^U = 2e_{ii}^T f_{ik}^T e_{kk}^T = e_{ik}^U.$$

If $k = i$ we have $4f_{ij}^T f_{ji}^T = (f_{ii}^T + f_{jj}^T)$, $e_{ij}^U e_{ji}^U = e_{ii}^T (e_{ii}^T + e_{jj}^T) e_{ii}^T = e_{ii}^T$. Hence in all cases we have (21), (24) and U is an automorphism of S . But $2f_{ij}^U = e_{ij}^U + e_{ji}^U = 2(f_{ii}^T f_{ij}^T f_{jj}^T + f_{jj}^T f_{ij}^T f_{ii}^T) = f_{ii}^T (2f_{ij}^T f_{jj}^T) + (2f_{jj}^T f_{ii}^T) f_{ii}^T = 2f_{ii}^T f_{ij}^T + 2f_{ij}^T f_{ii}^T = 2f_{ij}^T$, $f_{ij}^U = f_{ij}^T$ and we have proved that $a \rightarrow a^U$ is the given isomorphism of A and B .

Consider next the case where J is defined as in (3), (6). Then as we noted in the proof of Theorem 1 the intersection of M and S_J is the set of all $Ag_{11} + Ag_{22} = A = A'$ in M . By the proof above there is an isomorphism U of M with $\overline{M \cap S_J}$ such that $A^T = A^U$ for $A = A'$ in M . We may consider U as an automorphism of S leaving $g_{11}, g_{12}, g_{21}, g_{22}$ unaltered and may replace $(S_J)^T$ by the isomorphic special Jordan algebra $(S_J)^{TU^{-1}}$ and thus replace each A^T defined for $A = A'$ in M by A . Hence there is no loss of generality if we assume that $A = A^T$ for every symmetric A in S_J . We are also studying only the case $m > 1$.

The quantities

$$(25) \quad \begin{aligned} h_{ij} &= e_{ij}g_{11} + e_{ji}g_{22} & (i, j = 1, \dots, m) \\ f_{ij} &= (e_{ij} - e_{ji})g_{12} = -f_{ji} & (i \neq j; i, j = 1, \dots, m) \\ d_{ij} &= (e_{ij} - e_{ji})g_{21} = -d_{ji} & (i \neq j; i, j = 1, \dots, m) \end{aligned}$$

are in S_J and, by hypothesis, we have

$$(26) \quad (h_{ij} + h_{ji})^T = h_{ij} + h_{ji} \quad (i, j = 1, \dots, m).$$

Since by hypothesis $h_{ii}^T = h_{ii}$ and since T is an isomorphism between the special Jordan algebras S_J and S_J^T we have $[h_{ii} \cdot (h_{jj} \cdot a)]^T = h_{ii} \cdot (h_{jj} \cdot a^T)$. Apply this with $a = f_{ij}, d_{ij}, h_{ij}$ in turn and obtain

$$\begin{aligned}
 (27) \quad & 4h_{1i} \cdot (h_{jj} \cdot f_{ij}^T) = f_{ij}^T \\
 & 4h_{1i} \cdot (h_{jj} \cdot d_{ij}^T) = d_{ij}^T \quad i \neq j. \\
 & 4h_{1i} \cdot (h_{jj} \cdot h_{ij}^T) = h_{ij}^T
 \end{aligned}$$

This yields, by direct computation,

$$\begin{aligned}
 (28) \quad & f_{ij}^T = \alpha_{ij} f_{ij} + \beta_{ij} d_{ij} + \gamma_{ij} h_{ij} + \delta_{ij} h_{ji} \\
 & d_{ij}^T = \alpha'_{ij} f_{ij} + \beta'_{ij} d_{ij} + \gamma'_{ij} h_{ij} + \delta'_{ij} h_{ji} \\
 & h_{ij}^T = \alpha''_{ij} f_{ij} + \beta''_{ij} d_{ij} + \gamma''_{ij} h_{ij} + \delta''_{ij} h_{ji} \\
 & \quad (i, j = 1, \dots, m).
 \end{aligned}$$

But, by (9), $(f_{1i} \cdot (h_{1i} + h_{11}))^T = 0 = f_{1i}^T \cdot (h_{1i} + h_{11})$.

$(d_{1i} \cdot (h_{1i} + h_{11}))^T = 0 = d_{1i}^T \cdot (h_{1i} + h_{11})$ and we conclude that

$$\begin{aligned}
 (29) \quad & f_{1i}^T = \alpha_{1i} f_{1i} + \beta_{1i} d_{1i} + \gamma_{1i} h_{1i} - \delta_{1i} h_{11} \quad (i = 2, \dots, m). \\
 & d_{1i}^T = \alpha'_{1i} f_{1i} + \beta'_{1i} d_{1i} + \gamma'_{1i} h_{1i} - \delta'_{1i} h_{11}
 \end{aligned}$$

Since we have for $k \neq 1$, $2(f_{1i} \cdot (h_{ki} + h_{1k}))^T = 2f_{1i}^T \cdot (h_{ki} + h_{1k})$
 $= f_{1k}^T = \alpha_{1k} f_{1k} + \beta_{1k} d_{1k} + \gamma_{1k} h_{1k} - \delta_{1k} h_{k1} = \alpha_{1i} f_{1k} + \beta_{1i} d_{1k}$
 $+ \gamma_{1i} h_{1k} - \delta_{1i} h_{k1}$, and similar results for d_{1i} , we can write

$$\begin{aligned}
 (30) \quad & f_{1i}^T = \alpha_{1i} f_{1i} + \beta_{1i} d_{1i} + \gamma_{1i} h_{1i} - \delta_{1i} h_{11} \quad (i = 2, \dots, m). \\
 & d_{1i}^T = \alpha'_{1i} f_{1i} + \beta'_{1i} d_{1i} + \gamma'_{1i} h_{1i} - \delta'_{1i} h_{11}
 \end{aligned}$$

Observe that by (10) and (11), $f_{1i}^2 = d_{1i}^2 = (f_{1i}^T)^2 = (d_{1i}^T)^2 = 0$
 and hence

$$(31) \quad \alpha\beta + \gamma^2 = \alpha'\beta' + \gamma'^2 = 0.$$

If now $\alpha = 0$ then $\gamma = 0$, $\beta \neq 0$, $f_{1i}^T = \beta d_{1i}$. Let V be the automorphism of S

$$(32) \quad a \mapsto a^V = \alpha v a v^{-1}$$

with $v = g_{11} + g_{12} + \beta g_{21}$, which by direct computation leaves $h_{ij} + h_{ji}$ invariant and which has the further property that

$$(33) \quad (f_{11}^T)^V = f_{11} \quad (i = 2, \dots, m).$$

If $\alpha \neq 0$ we take v in (32) to be $v = g_{11} + \gamma g_{21} + \alpha g_{22}$, and we see that V leaves $h_{ij} + h_{ji}$ invariant, and again (33) is valid.

Thus we may assume

$$(34) \quad \begin{aligned} (h_{ij} + h_{ji})^T &= h_{ij} + h_{ji} & (i, j = 1, \dots, m), \\ f_{11}^T &= f_{11} & (i = 2, \dots, m). \end{aligned}$$

We observe that $2(f_{11} \cdot d_{11})^T = 2f_{11}^T \cdot d_{11}^T = 2f_{11} \cdot d_{11}^T = -(h_{11} + h_{11})^T = -h_{11} - h_{11}$ implies by (11) and (30), (31) that $\beta' = 1$, $\alpha' + \gamma'^2 = 0$. Let now W be an inner automorphism of $\mathfrak{S}$ generated by

$$(35) \quad w = g_{11} + g_{22} - \gamma' g_{12}.$$

Using direct computation we see that W leaves $(h_{ij} + h_{ji})$ and f_{11} invariant, and

$$(36) \quad (d_{11}^T)^W = d_{11} \quad (i = 2, \dots, m).$$

This result permits us to assume

$$(37) \quad \begin{aligned} (h_{ij} + h_{ji})^T &= h_{ij} + h_{ji} & (i, j = 1, \dots, m) \\ f_{11}^T &= f_{11} & (i = 2, \dots, m). \\ d_{11}^T &= d_{11} \end{aligned}$$

By (38) we may write

$$(38) \quad \begin{aligned} h_{11}^T &= \alpha_1 f_{11} + \beta_1 d_{11} + \gamma_1 h_{11} + \delta_1 h_{11} & (i = 2, \dots, m) \\ h_{11} &= \alpha'_1 f_{11} + \beta'_1 d_{11} + \gamma'_1 h_{11} + \delta'_1 h_{11} \end{aligned}$$

where, by (37),

$$(39) \quad \begin{aligned} \alpha'_1 &= -\alpha_1, & \beta'_1 &= -\beta_1, \\ \gamma_1 + \gamma'_1 &= \delta_1 + \delta'_1 = 1. \end{aligned}$$

By (9) and (37) we compute $2(h_{11} \cdot (h_{11} + h_{11}))^T = 2h_{11}^T \cdot (h_{11} + h_{11}) = (h_{11} + h_{11})^T = h_{11} + h_{11}$ and find

$$(40) \quad \gamma_1 + \delta_1 = 1$$

and, similarly,

$$(41) \quad \gamma'_1 + \delta'_1 = 1.$$

Using (9) and (37) we find $2(h_{11} \cdot (h_{k1} + h_{1k}))^T = 2h_{11}^T \cdot (h_{k1} + h_{1k})$
 $= h_{1k}^T = \alpha_1 f_{1k} + \beta_1 d_{1k} + \gamma_1 h_{1k} + \delta_1 h_{k1}$ for $k \neq 1$, 1 with similar
 results for h_{11}^T . Hence we may write

$$(42) \quad \begin{aligned} h_{11}^T &= \alpha f_{11} + \beta d_{11} + \gamma h_{11} + \delta h_{11} \quad (i = 2, \dots, m), \\ h_{11}^T &= \alpha' f_{11} + \beta' d_{11} + \gamma' h_{11} + \delta' h_{11} \\ \alpha' &= -\alpha, \quad \beta' = -\beta, \\ \gamma + \gamma' &= \delta + \delta' = \gamma + \delta = \gamma' + \delta' = 1. \end{aligned}$$

Since by (9) $h_{11}^2 = (h_{11}^T)^2 = h_{11}^2 = (h_{11}^T)^2 = 0$ we have

$$(43) \quad \alpha\beta - \gamma\delta = \alpha'\beta' - \gamma'\delta' = 0, \quad \gamma\delta = \gamma'\delta'.$$

If we compute $(h_{11} \cdot f_{11})^T = (h_{11} \cdot d_{11})^T = 0 = h_{11}^T \cdot f_{11} = h_{11}^T \cdot d_{11}$
 and use (10), (11), (42) and (43) we find

$$(44) \quad \begin{aligned} \alpha &= \beta = \alpha' = \beta' = 0, \\ \gamma\delta &= \gamma'\delta' = \gamma(1 - \gamma) = \gamma'(1 - \gamma') = 0 \end{aligned}$$

and therefore

$$(45) \quad \begin{array}{cc} \gamma = 0 & \gamma = 1 \\ \gamma' = 1 & \text{or} \quad \gamma' = 0 \end{array}$$

for $\gamma = 0$, $\gamma' = 0$ or $\gamma = \gamma' = 1$ would contradict the last line
 of (42). If $h_{11}^T = h_{11}$, $h_{11}^T = h_{11}$ we would have $h_{1k}^T = 2(h_{11} \cdot h_{1k})^T$
 $= 2h_{11} \cdot h_{k1} = h_{k1}$, $2(f_{11} \cdot h_{1k})^T = 0 = 2f_{11} \cdot h_{k1} = f_{1k}$ which is
 impossible for $k \neq 1$, 1. Therefore

$$(46) \quad h_{11}^T = h_{11}, \quad h_{11}^T = h_{11}.$$

Using now

$$\begin{aligned}
 (47) \quad & h_{ij}^T = -2f_{11}^T \cdot d_{ij}^T & (i, j = 1, \dots, m), \\
 & f_{ij}^T = -2f_{11}^T \cdot h_{j1}^T & (i \neq j; i, j = 1, \dots, m), \\
 & d_{ij}^T = -2d_{11}^T \cdot h_{j1}^T & (i \neq j; i, j = 1, \dots, m),
 \end{aligned}$$

we see that now

$$\begin{aligned}
 (48) \quad & h_{ij}^T = h_{ij} \\
 & f_{ij}^T = f_{ij} \\
 & d_{ij}^T = d_{ij}
 \end{aligned}$$

and this proves that there exists an inner automorphism R of S inducing the originally given isomorphism T of S_J , namely,
 $R = WVU$.

We may now prove

THEOREM 4. Let S be a simple associative algebra over its centrum F , J and K two involutions of S over F . Then each isomorphism T over F between S_J and S_K of order greater than one can be extended uniquely to an automorphism U over F of S . If $J = K$ then the automorphism U which is an extension of the given automorphism T of S_J is generated by a J -orthogonal element u of S , that is, $uu^J = \alpha$ in F .

For there exists a scalar extension L of the centrum F of S such that S_L is a total matrix algebra, J and K can be extended to S_L by Lemma 1, and the extensions $(S_J)_L$ and $(S_K)_L$ will be isomorphic. In S_L both J and K are cogredient to one of the involutions J_0 above, for otherwise, by Theorem 1, S_J and S_K would not have the same order as linear spaces over F . By the above argument the isomorphism between $(S_L)_J$ and $(S_L)_K$ can be extended to an automorphism of S_L which induces the original isomorphism T between S_J and S_K and hence induces an automorphism of $\overline{S_J} = \overline{S_K} = S$.

The uniqueness follows from Theorem 1. For if $a_1, \dots, a_s$ is a basis of S_J over F then every element a of S has the form

$a = \sum \alpha a_{1_1} \dots a_{1_k}$ for α in F and if U and V are two extensions of T then $a^U = \sum \alpha a_{1_1}^T \dots a_{1_k}^T = a^V$, $U = V$.

If $J = K$ we have, for each a of S , by Theorem 1, $a^{JU} = \sum \alpha a_{1_k}^T \dots a_{1_1}^T = a^{UJ}$, $a^{JU} = ua^J u^{-1} = a^{UJ} = (u^{-1})^J a^J u^J$ and hence $uu^J = \alpha$ in F . This completes the proof of the theorem.

Consider now two simple associative algebras S^1 and S^2 over their centra F , and J and K involutions of S^1 and S^2 respectively. Assume also that S_J^1 and S_K^2 are isomorphic. There exists a scalar extension field L of F such that J in S_L^1 is cogredient to one of the involutions J_0 above, and at the same time K in S_L^2 is cogredient to the same J_0 for otherwise S_J^1 and S_K^2 could not be isomorphic. Thus by Lemma 5, $(S_J^1)_L$ is isomorphic to $(S_{J_0}^1)_{J_0}$ and so is $(S_K^2)_K$. The isomorphism between $(S_L^1)_J$ and $(S_L^2)_J$ thus induces an automorphism of $(S_{J_0}^1)_{J_0}$ which by Theorem 4 can be extended to an automorphism U of S_L^1 which will induce the desired isomorphism between $\overline{S_J^1} = S^1$ and $\overline{S_K^2} = S^2$. Thus we have proved the

COROLLARY. Let S^1 and S^2 be two simple associative algebras over the same centrum F , J and K involutions over F of S^1 and S^2 respectively. Then each isomorphism T between S_J^1 and S_K^2 of order greater than one can be extended to an isomorphism U between $\overline{S_J^1} = S^1$ and $\overline{S_K^2} = S^2$.

Lemma 5 and Theorem 4 imply

THEOREM 5. Let S be a simple associative algebra over its centrum F . Then given two involutions J and K of S over F S_J and S_K are isomorphic if and only if J and K are cogredient, provided S_J and S_K have order greater than one.

If J and K are cogredient then S_J and S_K are isomorphic by Lemma 5. Suppose that S_J and S_K are isomorphic. Then the isomorphism T between S_J and S_K can be extended, by Theorem 4, to

an automorphism U of S . By Theorem 1, every a of S is of the form $a = \sum \alpha a_{1_1} \cdots a_{1_k}$ for α in F and the a_i are a basis of S_J over F . Then $a^J U = \sum \alpha a_{1_k}^T \cdots a_{1_1}^T = a^{UK}$, $JU = UK$, J and K are cogredient.

Two J -orthogonal elements g and f of an associative simple algebra S generate the same automorphism of S_J if and only if $g = \alpha f$ for some α in F , for $gag^{-1} = faf^{-1}$ for all a in S_J implies the same equation for all a in S by Theorem 1, provided that the order of S_J be greater than one; but $gag^{-1} = faf^{-1}$ for all a of S implies that $g = \alpha f$ for some α in F . We have proved

THEOREM 6. Let S be a simple associative algebra over its centrum F , J be an involution over F of S such that the order of S_J is greater than one. Then the group of automorphisms of S_J over F is isomorphic to the group H^*/F^* where H^* is the multiplicative group of all regular J -orthogonal elements of S and F^* is the multiplicative group of F .

7. Classification of the algebras S_J over special fields.

Lemma 4 and Theorem 5 imply that if S is a simple associative algebra over its centrum F , the classification of its special Jordan subalgebras S_J is equivalent to the classification of J -symmetric and J -skew elements of S as to J -congruence. This depends on the structure of the ground field F . If F is an algebraically closed field then every J is cogredient either to the operation of transposition of matrices (for we may, of course, consider S as a total matrix algebra over F), or to J as defined by (3) and (6)⁶.

If F is a real closed field it is necessary to distinguish the case where S is a total matrix algebra over F from that where

⁶Cf. A. A. Albert, Modern Higher Algebra, pp. 108, 110.

S is a total matrix algebra with quaternionic elements. In the first case if J denotes matrix transposition the J -symmetric elements are J -congruent to

$$(49) \quad \left\| \begin{array}{cc} I_p & 0 \\ 0 & -I_{n-p} \end{array} \right\|$$

where we take $p \geq 1/2n$ since we are permitted to multiply by real factors⁷. There are $|1/2n| + 1$ types of nonisomorphic special Jordan algebras in this case. In the case of J -skew elements there is only one possibility, namely, the degree of the algebra S is an even number $n = 2m$ and J is given by (3) and (6)⁸.

In the case of real quaternionic matrices it is best to take for J the correspondence $a \rightarrow \bar{a}'$ where a' is the transpose of a and $\bar{a}$ is the matrix of the conjugate elements of a . In the symmetric case we again obtain (49)⁹ and there are $|1/2n| + 1$ types of nonisomorphic special Jordan algebras. In the skew case we obtain one type given by

$$(50) \quad \left\| \begin{array}{c} t \\ t \\ \cdot \\ \cdot \\ \cdot \\ t \end{array} \right\|$$

where t is a quaternion such that $\bar{t} = -t$.¹⁰

8. The main theorem. While it is easy to see what happens to an algebra upon extension of its base field it is quite difficult to determine the structure of an algebra from that of one of

⁷Cf. A. A. Albert, loc. cit., p. 114.

⁸Cf. A. A. Albert, loc. cit., p. 108.

⁹Cf. N. Jacobson, Duke Journal of Mathematics, 4 (1938), p. 546.

¹⁰Cf. N. Jacobson, loc. cit., p. 546.

its scalar extensions. Thus, if A over F is a special Jordan algebra S_J for some associative algebra S over F with an involution J , it is clear, by Lemma 1, that for every extension field L of F , $A_L = (S_J)_L = (S_L)_J$ is again a special Jordan algebra of the same kind. We shall show that in certain cases we can prove a converse theorem, namely, that if given an algebra A over F , there exists a scalar extension field L of F such that A_L is isomorphic to a special Jordan algebra S_J for S over L , then there exists an associative algebra T over F with an involution J such that A is isomorphic to T_J .

Consider an algebra A over F and assume that there exists a normal separable scalar extension field L of F such that A_L is isomorphic to S_J where S is a total matrix algebra over L , and J is an involution of S over L cogredient either to transposition J_1 or to $J = J_2$ as defined by (3) and (6). If s is in S , we can write s as the matrix $\| s_{ij} \|$ with s_{ij} in L and if t is in Galois group G of L over F , we can introduce the following linear map of S onto itself: $s = \| s_{ij} \| \rightarrow s^t = \| s_{ij}^t \|$ where s_{ij}^t is the map of s_{ij} under t of G . This map is an automorphism of S over F . It is clear that t commutes with J_1 . If s is written as in (3) we have

$$(51) \quad s^t = A^t s_{11} + B^t s_{12} + C^t s_{21} + D^t s_{22}$$

where the matrices A^t, B^t, C^t, D^t have the obvious meaning as defined above. Again it is clear that t and J_2 commute. Suppose that $a_1, \dots, a_m$ is a basis of A over F , and hence of A_L over L . Denote by $b_1, \dots, b_m$ the corresponding matrices of S forming a basis of S_J over L . The elements a in S_J have the form $a = \sum c_i b_i$ with c_i in L . Then $a \rightarrow a^t = \sum c_i b_i^t$ defines an automorphism of S_J over L which, by Theorem 4, can be extended to an

automorphism of S over L which we shall also denote by T . Again we have $TJ_1 = J_1T$, $TJ_2 = J_2T$.

Consider the set B consisting of all s in S such that $s^t = s^T$ for all t of G . B is an associative algebra over F as it contains with any a , b also the elements ab and $\alpha a + \beta b$ for α and β in F , and if γ is in L not in F , and a is in B , then γa is not in B , for $(\gamma a)^t = (\gamma a)^T = \gamma^t a^t = \gamma a^T = \gamma a^t$ and $\gamma = \gamma^t$ for all t in G , γ is in F . Furthermore, the mappings J_1 or J_2 of S induce mappings K_1 or K_2 of B , because t and T commute with J_1 and J_2 as we have seen above. It is also clear that J_1 is an extension of K_1 . Moreover B contains the b_i and hence B_L contains all expressions $\sum \alpha b_{i_1} \cdots b_{i_k}$, α in L , which, by Theorem 1, constitute S (we may, of course, assume that the order of S_J is greater than one). Thus $B_L \supseteq S$, and since obviously $S \supseteq B_L$, we have $S = B_L$, and B is simple over its centrum F . Let M be the special Jordan algebra of matrices whose basis is $b_1, \dots, b_m$ over F , M is isomorphic to A . Since the b_i are in B_K (for $K = K_1$ or K_2) we have $B_K \supseteq M$. Consider now c in B_K . First we have $c^K = c$ which means that c is in S_J and thus $c = \sum c_i b_i$ for c_i in L . But we also have $c^t = c^T$ and hence $\sum c_i b_i^t = \sum c_i^t b_i^t$. Now the b_i^t are linearly independent because of the linear independence of the b_i , and we therefore have $c_i = c_i^t$ for all t in G , c_i is in F , c is in M , $B_K \supseteq M$ and A is isomorphic to B_K .

We are now in position to prove the main theorem.

THEOREM 7. Let A be an algebra over a nonmodular field F , and let L be a finite scalar extension field of F . Then if A_L is isomorphic to a special Jordan algebra S_J for S simple over its centrum L and having an involution J over L , there exists an associative simple algebra B over its centrum F with an involution

K such that $B_L = S$, J is the extension of K and A is isomorphic to B_K .

For there exists a finite scalar extension field L' of L such that L' is normal separable over F , $S_{L'}$ is a total matrix algebra, and the extension of J to $S_{L'}$ is cogredient in L' either to transposition or to J as defined by (3) and (6). Then $A_{L'}$ is isomorphic to $(S_{L'})_J$ and by the above argument there exists an associative simple algebra B over F with an involution K such that A is isomorphic to B_K , $B_{L'} = S_{L'}$, J is an extension of K . It is clear that this algebra B has the properties mentioned in the theorem.

VITA

Gerhard Karl Kalisch was born on December 21, 1914 in Breslau, Germany, where he received his grammar school and high school education. Upon graduation from the "Gymnasium" in 1933, he went to France where he was engaged in agricultural activities. He came to the United States in 1937 at which time he was granted a private fellowship to the State University of Iowa at Iowa City, Iowa. There he received his Bachelor of Arts degree in 1938 in mathematics, and the Master of Science degree in 1939, also in mathematics. He attended the University of Chicago continuously from October 1939 to September 1941, doing graduate work under Professors Albert, Bliss, Graves, Hestenes, Lane, MacLane, Reid, Schilling and Steenrod. To Professor Albert he wishes to express his gratitude and sincere appreciation for suggestions and helpful criticism during the preparation of this thesis.

